EXACT D.E.

Definition → A D.E. of first order and of first degree of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be an exact D.E. if there exists a function $u(x,y)$ such that the differential $M(x,y)dx + N(x,y)dy$ is, without multiplication by any factor, expressible in the form du.

i.e., if the equation $M dx + N dy = 0$ is exact, then $\xrightarrow{}$ ①

$du = M\,dx + N\,dy = 0.$

The g.s. of ① is then $u(x,y) = C$, C being an arbitrary constant.

THEOREM: A Necessary and Sufficient condition for the D.E. $M dx + N dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

PROOF:– Condition is necessary:

Let the D.E. $M dx + N dy = 0$ be an exact DE.

Hence $\exists$ a function $u(x,y)$ s.t.

$$M\,dx + N\,dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

$$\Rightarrow \quad M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

Assuming $u(x,y)$ has continuous partial derivatives, we have $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. [Necessary condition]

Condition is sufficient:

Let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ be held.

Let $V = V(x,y)$ s.t. $\frac{\partial V}{\partial x} = M \Rightarrow V = \int M\,dx$.
(y constant)

Now $\frac{\partial M}{\partial y} = \frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y}$ [assuming $V_{yx} = V_{xy}$].
i.e. $V(x,y)$ has continuous partial derivatives.

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, we have

$\dfrac{\partial N}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial y}\right)$, which on integration partially w.r.t. x when y is treated as constant, gives

$$N = \frac{\partial V}{\partial y} + f(y) = \frac{\partial V}{\partial y} + \phi'(y), \text{ where } \phi'(y) = f(y), \text{ say.}$$

Thus
$$\begin{aligned} M\,dx + N\,dy &= \frac{\partial V}{\partial x}dx + \left[\frac{\partial V}{\partial y} + \phi'(y)\right]dy \\ &= \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \phi'(y)\,dy \\ &= dV + d\{\phi(y)\} = d\{V + \phi(y)\} \\ &= du, \text{ where } u = V + \phi(y). \end{aligned}$$

$\therefore$ $M\,dx + N\,dy = 0$ is an exact D.E.

Examples of Exact Differentials:

(1) $x\,dy + y\,dx = d(xy)$, (2) $x\,dx + y\,dy = d\left(\dfrac{x^2+y^2}{2}\right)$.

(3) $\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$, (4) $\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$.

(5) $\dfrac{x\,dy + y\,dx}{xy} = d\{\log(xy)\}$. (6) $\dfrac{y\,dx - x\,dy}{xy} = d\left(\log\dfrac{x}{y}\right)$.

(7) $\dfrac{x\,dy - y\,dx}{x^2+y^2} = \dfrac{(x\,dy - y\,dx)/x^2}{(x^2+y^2)/x^2} = \dfrac{d\left(\frac{y}{x}\right)}{1+\left(\frac{y}{x}\right)^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$.

(8) $\dfrac{y\,dx - x\,dy}{x^2+y^2} = \dfrac{(y\,dx - x\,dy)/y^2}{\left(\frac{x}{y}\right)^2+1} = d\left(\tan^{-1}\dfrac{x}{y}\right)$.

(9) $\dfrac{x\,dx + y\,dy}{x^2+y^2} = \dfrac{\frac{1}{2}d(x^2+y^2)}{x^2+y^2} = \dfrac{1}{2}d\{\log(x^2+y^2)\}$.

(10) $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2+y^2} = d\left\{\dfrac{1}{2}(x^2+y^2)\right\} + d\left(\tan^{-1}\dfrac{y}{x}\right)$
$= d\left\{\dfrac{1}{2}(x^2+y^2) + \tan^{-1}\left(\dfrac{y}{x}\right)\right\}$.

Solve the exact D.E.

Ex. 1. $(4x^3y^3 - 2xy)\,dx + (3x^4y^2 - x^2)\,dy = 0 \longrightarrow (1)$

Comparing the given D.E. with $M\,dx + N\,dy = 0$,

we have $M = 4x^3y^3 - 2xy$. $N = 3x^4y^2 - x^2$.

$\frac{\partial M}{\partial y} = 12x^3y^2 - 2x$, $\frac{\partial N}{\partial x} = 12x^3y^2 - 2x$.

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ The D.E. (1) is exact.

Therefore, $\exists$ a function $u(x,y)$ s.t.

$(4x^3y^3 - 2xy)\,dx + (3x^4y^2 - x^2)\,dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

$\Rightarrow \frac{\partial u}{\partial x} = 4x^3y^3 - 2xy \rightarrow (2)$ and $\frac{\partial u}{\partial y} = 3x^4y^2 - x^2 \longrightarrow (3)$.

Integrating (2) partially w.r.t. x, we obtain

$\underset{(y\text{ constant})}{\int \frac{\partial u}{\partial x}\,dx} = \underset{(y\text{ constant})}{\int (4x^3y^3 - 2xy)\,dx} = 4\frac{x^4}{4}y^3 - 2\frac{x^2}{2}\cdot y + f(y)$

$\Rightarrow u = x^4y^3 - x^2y + f(y) \longrightarrow (4)$, $f(y)$ is an arbitrary function of y.

Now differentiating (4) partially w.r.t. y, we get

$\frac{\partial u}{\partial y} = \cancel{3x^4y^2} - \cancel{x^2} + f'(y) = \cancel{3x^4y^2} - \cancel{x^2}$ [from (3)]

$\Rightarrow f'(y) = 0 \Rightarrow f(y) = c_1$ [by integration]

where c_1 is an arbitrary constant.

We have the g.s. of the exact DE (1) is given by $du = 0 \Rightarrow u(x,y) = \text{Constant} = c_2$ (say)

$\therefore$ From (4), we get

$u = x^4y^3 - x^2y + c_1 = c_2$

or, $x^4y^3 - x^2y = c_2 - c_1 = c$ (say).

$\therefore$ g.s. $\boxed{x^4y^3 - x^2y = c}$

Ex. ② Solve the D.E. $(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy=0$ $\longrightarrow$ ①.

Comparing the given D.E. ① with $M(x,y)dx+N(x,y)dy=0$, we have $M=x^2-4xy-2y^2$; $N=y^2-4xy-2x^2$.

$\therefore \frac{\partial M}{\partial y}=-4x-4y$; $\frac{\partial N}{\partial x}=-4y-4x$.

$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow$ The D.E. ① is exact.

Therefore, $\exists$ a function $u(x,y)$ s.t.

$(x^2-4xy-2y^2)dx+(y^2-4xy-2x^2)dy = du = \frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$.

$\Rightarrow \frac{\partial u}{\partial x}=x^2-4xy-2y^2 \longrightarrow$ ②, $\frac{\partial u}{\partial y}=y^2-4xy-2x^2 \longrightarrow$ ③

Integrating ② partially w.r.t. x, we obtain

$\underset{(y\text{ constant})}{\int \frac{\partial u}{\partial x}dx} = \underset{(y\text{ constant})}{\int (x^2-4xy-2y^2)\,dx} = \frac{x^3}{3}-4\frac{x^2}{2}y-2xy^2+f(y)$

or, $u=\frac{x^3}{3}-2x^2y-2xy^2+f(y)$, $f(y)$ is an arbitrary function of y. $\longrightarrow$ ④.

Now differentiating ④ partially w.r.t. y, we get

$\frac{\partial u}{\partial y}=-\cancel{2x^2}-\cancel{4xy}+f'(y)=y^2-\cancel{4xy}-\cancel{2x^2}$ [from ③]

$\Rightarrow f'(y)=y^2 \Longrightarrow f(y)=\frac{y^3}{3}+c_1$ [by integrating]

$\therefore$ From ④, $u(x,y)=\frac{x^3}{3}-2x^2y-2xy^2+\frac{y^3}{3}+c_1$

The g.s. of ① is given by $du=0 \Rightarrow u(x,y)=$ Constant

$\therefore$ The g.s. is given by:

$$\frac{x^3}{3}-2x^2y-2xy^2+\frac{y^3}{3}+c_1=c_2$$

or, $\underline{x^3-6x^2y-6xy^2+y^3=c}$, where $c=3(c_2-c_1)$.

where, c_1, c_2, c are arbitrary constants.

Ex. 3. Solve the D.E. $(1+6y^2-3x^2y)\frac{dy}{dx}=3xy^2-x^2$.

The given D.E. can be expressed in the form $M(x,y)\,dx+N(x,y)\,dy=0$ as

$$(3xy^2-x^2)\,dx+(3x^2y-6y^2-1)\,dy=0 \longrightarrow (1)$$

On comparison, we get

$M=3xy^2-x^2$ and $N=3x^2y-6y^2-1$

$\therefore \frac{\partial M}{\partial y}=6xy$; $\frac{\partial N}{\partial x}=6xy$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow \text{The D.E. (1) is Exact.}$$

Therefore, $\exists$ a function $u(x,y)$ s.t.

$$(3xy^2-x^2)\,dx+(3x^2y-6y^2-1)\,dy=du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy.$$

$$\Rightarrow \frac{\partial u}{\partial x}=3xy^2-x^2 \rightarrow (2) \quad ; \quad \frac{\partial u}{\partial y}=3x^2y-6y^2-1 \longrightarrow (3)$$

Integrating (2) partially w.r.t. x, we get

$$\int \frac{\partial u}{\partial x}dx = \int (3xy^2-x^2)\,dx = \frac{3x^2y^2}{2}-\frac{x^3}{3}+f(y)$$

(y constant) (y constant)

$$\Rightarrow u(x,y)=\frac{3x^2y^2}{2}-\frac{x^3}{3}+f(y), \quad f(y) \text{ is an arbitrary function of } y. \longrightarrow (4)$$

Differentiating (4) partially w.r.t. y, we get

$$\frac{\partial u}{\partial y}=\cancel{3x^2y}+f'(y)=\cancel{3x^2y}-6y^2-1 \quad [\text{from (3)}]$$

$$\Rightarrow f'(y)=-6y^2-1$$

Integrating, $\int f'(y)\,dy=\int(-6y^2-1)\,dy+c_1$

$$\Rightarrow f(y)=-2y^3-y+c_1$$

$$\therefore u(x,y)=\frac{3}{2}x^2y^2-\frac{x^3}{3}-2y^3-y+c_1$$

The g.s. of (1) is given by $du=0$

$$\Rightarrow u(x,y)=c_2$$

$$\Rightarrow \frac{3}{2}x^2y^2-\frac{x^3}{3}-2y^3-y+c_1=c_2$$

$$\Rightarrow \underline{9x^2y^2-2x^3-12y^3-6y=c.}$$

Where $c=6(c_2-c_1)$, is an arbitrary constant.

Ex. 4. Solve the D.E. $(y^2e^{xy^2}+4x^3)\,dx + (2xye^{xy^2}-3y^2)\,dy = 0$. ⟶ ①

Comparing the given DE with $M(x,y)\,dx + N(x,y)\,dy = 0$,

we get $M = y^2e^{xy^2}+4x^3$; $N = 2xye^{xy^2}-3y^2$.

$\therefore \frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 \cdot 2xye^{xy^2}$; $\frac{\partial N}{\partial x} = 2ye^{xy^2} + 2xy \cdot y^2e^{xy^2}$

$= e^{xy^2}(2y+2xy^3)$ $\qquad = e^{xy^2}(2y+2xy^3)$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ the D.E. is exact.

Therefore, $\exists$ some function $u(x,y)$ s.t.

$(y^2e^{xy^2}+4x^3)\,dx + (2xye^{xy^2}-3y^2)\,dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$.

$\Rightarrow \frac{\partial u}{\partial x} = y^2e^{xy^2}+4x^3$ ⟶ ② and $\frac{\partial u}{\partial y} = 2xye^{xy^2}-3y^2$ ⟶ ③

Integrating ② partially w.r.t. x, we obtain

$\underset{(y\text{ constant})}{\int \frac{\partial u}{\partial x}dx} = \underset{(y\text{ constant})}{\int (y^2e^{xy^2}+4x^3)\,dx} = y^2 \cdot \frac{1}{y^2}e^{xy^2} + 4 \cdot \frac{x^4}{4} + f(y)$

$u(x,y) = e^{xy^2} + x^4 + f(y)$ ⟶ ④, $f(y)$ being an arbitrary function.

Differentiating ④ partially w.r.t. y, we get

$\frac{\partial u}{\partial y} = x \cdot 2y\,e^{xy^2} + f'(y) = 2xye^{xy^2} - 3y^2$ [from ③]

$\Rightarrow f'(y) = -3y^2 \Rightarrow \int f'(y)\,dy = -3\int y^2\,dy + c_1$

$\Rightarrow f(y) = -y^3 + c_1$

$\therefore$ From ④, $u(x,y) = e^{xy^2} + x^4 - y^3 + c_1$

The g.s. of ① is given by $du = 0 \Rightarrow u(x,y) = c_2$

$\Rightarrow e^{xy^2} + x^4 - y^3 + c_1 = c_2$

$\Rightarrow e^{xy^2} + x^4 - y^3 = c$ $[c = c_2 - c_1]$.

c being an arbitrary constant.